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On an Inequality of Ky Fan

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1. INTRODUCTION

In the famous book "Inequalities" by E. F. Beckenbach and R. Bellman one can find the following "unpublished result due to Ky Fan" [1, p. 5]:

THEOREM. *If* $x_i \in (0, 1/2]$, $i = 1, ..., n$, *then*

$$\left[\prod_{i=1}^{n} x_i \Big/ \prod_{i=1}^{n} (1-x_i)\right]^{1/n} \leqslant \sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n} (1-x_i), \tag{1}$$

with equality only if $x_1 = \cdots = x_n$.

This inequality "can be established by forward and backward induction" [1, p. 5]—a method used by A. L. Cauchy to prove the inequality between the arithmetic and the geometric means.

In [4] Levinson has published an elegant generalization of (1):

THEOREM. *Let* $x_i \in (0, a]$, *and* $p_i > 0$, $i = 1, ..., n$, *with* $P_n = \sum_{i=1}^{n} p_i$. *If the function* f *has a nonnegative third derivative on* $(0, 2a)$, *then*

$$\sum_{i=1}^{n} f(x_i)\, p_i/P_n - f\left(\sum_{i=1}^{n} x_i\, p_i/P_n\right) \leqslant \sum_{i=1}^{n} f(2a-x_i)\, p_i/P_n - f\left(\sum_{i=1}^{n} (2a-x_i)\, p_i/P_n\right). \tag{2}$$

If the third derivative of f *is positive on* $(0, 2a)$, *then equality holds in* (2) *only if* $x_1 = \cdots = x_n$.

Indeed, if we set $a = 1/2$, $p_1 = \cdots = p_n = 1/n$ and $f = \ln$ in (2), then we obtain Fan's inequality (1).

Popoviciu [5] and Bullen [2] have given remarkable extensions of (2).

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In a short note published in 1980 Wang [7] proved the following inequality of Rado type by using elementary analysis:

$$P_n[A_n(G_n+G'_n)-G_n] \geqslant P_{n-1}[A_{n-1}(G_{n-1}+G'_{n-1})-G_{n-1}], \tag{3}$$

where

$$A_n=\sum_{i=1}^{n} x_i\, p_i/P_n, \qquad G_n=\prod_{i=1}^{n} x_i^{p_i/P_n},$$

$$A'_n=\sum_{i=1}^{n} (1-x_i)\, p_i/P_n, \qquad G'_n=\prod_{i=1}^{n} (1-x_i)^{p_i/P_n},$$

$0<x_i \leqslant 1/2$, $i=1, ..., n$, and $P_n=\sum_{i=1}^{n} p_i$ with positive weights $p_1, ..., p_n$.

A simple calculation yields that

$$G_n \leqslant A_n(G_n+G'_n) \tag{4}$$

follows inductively from (3). Since $A_n+A'_n=1$, (4) is equivalent to

$$G_n/G'_n \leqslant A_n/A'_n, \tag{5}$$

which is a little more general form of (1).

Wang called (5) "a Ky Fan inequality of the complementary A-G type" [7, p. 502], where A-G is an abbreviation of arithmetic and geometric.

In [8] inequality (5) has been established by the interesting functional equation approach of dynamic programming.

The aim of this paper is to give an affirmative answer to the following question asked by Wang: "Are there more proofs of inequality (1) in addition to the one by Levinson ··· and the original, unpublished one?" [7, p. 502].

In Section 2 we present two new short and simple proofs of inequality (1). Both proofs have in common that the main ideas have been taken from not so well known approaches of the inequality between the arithmetic and the geometric means. This displays the "strong resemblance between the Ky Fan inequality (1) and the A-G inequality" [7, p. 502].

2. Two New Proofs of Fan's Inequality

Let $x_i \in (0, 1/2]$, $i=1, ..., n$. In this section we denote by A_n and G_n (resp. A'_n and G'_n) the arithmetic and geometric means of $x_1, ..., x_n$ (resp. $1-x_1, ..., 1-x_n$), i.e.,

$$A_n = \frac{1}{n} \sum_{i=1}^{n} x_i, \qquad G_n = \left[\sum_{i=1}^{n} x_i \right]^{1/n},$$

$$A'_n = \frac{1}{n} \sum_{i=1}^{n} (1 - x_i), \qquad G'_n = \left[\prod_{i=1}^{n} (1 - x_i) \right]^{1/n}.$$

The first proof of (1) which we want to give has been motivated by a short paper by Chong [3] published in 1976.

For $n = 2$, (1) is equivalent to

$$(x_1 - x_2)^2 (1 - x_1 - x_2) \geqslant 0, \tag{6}$$

which is true for $x_1, x_2 \in (0, 1/2]$ with equality only if $x_1 = x_2$. Now we assume that (1) is valid for $n - 1$. Without loss of generality we may write

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n, \qquad x_1 < x_n.$$

Then we have

$$\frac{A_n}{A'_n} = \frac{(x_1 + x_n - A_n) + \sum_{i=2}^{n-1} x_i}{1 - (x_1 + x_n - A_n) + \sum_{i=2}^{n-1} (1 - x_i)}$$

and by the induction hypothesis we obtain

$$\left(\frac{A_n}{A'_n} \right)^{n-1} \geqslant \frac{(x_1 + x_n - A_n) \prod_{i=2}^{n-1} x_i}{(1 - (x_1 + x_n - A_n)) \prod_{i=2}^{n-1} (1 - x_i)}. \tag{7}$$

A short calculation yields that the inequality

$$\frac{(x_1 + x_n - A_n) A_n}{(1 - (x_1 + x_n - A_n)) A'_n} > \frac{x_1 x_n}{(1 - x_1)(1 - x_n)} \tag{8}$$

is equivalent to

$$(1 - x_1 - x_n)(A_n - x_1)(x_n - A_n) > 0. \tag{9}$$

Since

$$0 < x_1 < A_n < x_n \leqslant 1/2$$

inequality (9) is valid and we conclude from (7) and (8)

$$(A_n / A'_n)^n > (G_n / G'_n)^n,$$

which we had to show.

The second approach of (1) has been inspired by Shisha's proof of the A.M.-G.M. inequality, which "appears to be somewhat simpler than the differential proof of Liouville" [6, p. 305]. Again we suppose

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n, \qquad x_1 < x_n,$$

then we have

$$A_{n-1} < x_n.$$

We assume that (1) holds for $n-1$.

To establish (1), we consider the function f defined by

$$f\colon (0, 1) \to \mathbb{R}$$

$$f(x) = (n-1) \ln(G'_{n-1}/G_{n-1}) + n \ln(x + (n-1)\, A_{n-1})$$
$$- n \ln(1 - x + (n-1)\, A'_{n-1}) + \ln(1/x - 1).$$

Differentiation of f yields

$$f'(x) \frac{x(1-x)}{n(n-1)} [(n-1) A_{n-1} + x][(n-1)\, A'_{n-1} + 1 - x]$$
$$= \left[1 - x - \frac{1}{n} ((n-1)\, A_{n-1} + x)\right] [x - A_{n-1}].$$

If $x \leqslant 1/2$, then we have

$$1 - x - \frac{1}{n} ((n-1)\, A_{n-1} + x) > 0.$$

Therefore we get

$$f'(x) > 0 \qquad \text{and} \qquad f(x) > f(A_{n-1}) \qquad \text{for} \quad A_{n-1} < x \leqslant 1/2,$$

and by the induction hypothesis we obtain

$$f(A_{n-1}) = (n-1) \ln(G'_{n-1}/G_{n-1}) - (n-1) \ln(A'_{n-1}/A_{n-1}) \geqslant 0.$$

Hence

$$f(x_n) > f(A_{n-1}) \geqslant 0,$$

which completes our second proof of Fan's inequality.

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